MATH 245 S22, Exam 1 Solutions

1. Carefully define the following terms: composite, contrapositive.

   Let n be an integer with $n \geq 2$. We call n composite if there exists an integer a with $1 < a < n$ and $a|n$. For any propositions p, q, the contrapositive of conditional proposition $p \to q$ is the proposition $(\neg q) \to (\neg p)$.

2. Carefully state the following theorems: De Morgan's Law theorem (for propositions), Double Negation semantic theorem.

   The De Morgan's Law theorem states: for any propositions p, q, we have $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$ and $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$. The Double Negation semantic theorem states: for any proposition p, we have $p \equiv \neg\neg p$.

3. State the Disjunctive Syllogism semantic theorem, and prove it without truth tables. (you may use any other semantic theorems we have proved)

   The Disjunctive Syllogism semantic theorem states: For any propositions p, q, we must have $p \vee q, \neg q \vdash p$. All correct proofs must use the following proof structure (by definitions of $\forall, \vdash$):

   Let p, q be arbitrary. Assume that $p \vee q$ and $\neg q$ are both true.
   (middle part)
   Hence we must have p is true.

   METHOD 1 for middle part: Applying Double Negation, we must have $p \vee \neg(\neg q)$ is true. Applying Conditional Interpretation, we must have $(\neg q) \to p$ is true. Apply Modus Ponens to $(\neg q) \to p$ and $\neg q$.

   METHOD 2 for middle part: Because $p \vee q$ is true, there are two cases. If p is true, then we are happy. If instead q is true, that's impossible since we also know that $\neg q$ is true. So this second case cannot happen.

   NOTE: If you instead stated the theorem as: $p \vee q, \neg p \vdash q$, that is perfectly fine too.

4. Let $a, b, c \in \mathbb{Z}$ with $a \leq b$ and $0 \leq c$. Prove that $ac \leq bc$.

   Because $a \leq b$, we must have $b - a \in \mathbb{N}_0$. Because $c \in \mathbb{Z}$ and $0 \leq c$, we must have $c \in \mathbb{N}_0$. The product of two whole numbers is a whole number, so $(b - a)c \in \mathbb{N}_0$. Multiplying out, we get $bc - ac \in \mathbb{N}_0$, so $ac \leq bc$.

   SURPRISINGLY COMMON ERROR: Some students assumed $ac \leq bc$ is true, then worked toward $a \leq b$. This is not the goal here! $p \to q$ is not logically equivalent to $q \to p$.

5. Let $a, b, c \in \mathbb{Z}$ with $a|b$. Prove that $ac|bc^2$.

   Because $a|b$, there is some integer k with $ak = b$. Multiply both sides by c^2, getting $akc^2 = bc^2$. Because kc^2 is an integer, we must have $a|bc^2$.

   COMMON ERRORS: (1) The moment you try to use division, everything from that point on is incorrect. (2) "$m|n$" is a proposition, not a number.

6. Simplify the proposition $\neg((q \to p) \wedge (q \to r))$ as much as possible, where only basic propositions are negated. Be sure to justify each step.

   Applying De Morgan's Law, we get $(\neg(q \to p)) \vee (\neg(q \to r))$. Applying Thm 2.16 (negated conditional interpretation) twice, we get $(q \wedge \neg p) \vee (q \wedge \neg r)$. We can stop here, but it will be nicer if we apply distributivity (in reverse) to get $q \wedge ((\neg p) \vee (\neg r))$.

7. Simplify the proposition $\neg\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ \forall z \in \mathbb{Z}\ \ (x < y) \to (z \leq y)$ as much as possible, where only basic propositions are negated. Be sure to justify each step. DO NOT TRY TO PROVE OR DISPROVE THE RESULT, ONLY SIMPLIFY.

We first pull the negation into the quantifiers: $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ \exists z \in \mathbb{Z}\ \neg((x < y) \to (z \leq y))$. We then apply Thm 2.16 (negated conditional interpretation), getting: $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ \exists z \in \mathbb{Z}\ \ (x < y) \wedge \neg(z \leq y)$. We now rewrite the negation of the inequality as: $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ \exists z \in \mathbb{Z}\ \ (x < y) \wedge (z > y)$. We can stop here, but it will be nicer if we combine the inequalities as: $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ \exists z \in \mathbb{Z}\ \ x < y < z$.

8. Prove that $\oplus$ is associative. That is, prove:
For all propositions p, q, r, we must have $(p \oplus q) \oplus r \equiv p \oplus (q \oplus r)$.

In the truth table below, the fifth and seventh columns agree; hence the desired logical equivalence is proved.

p	q	r	$p \oplus q$	$(p \oplus q) \oplus r$	$q \oplus r$	$p \oplus (q \oplus r)$
T	T	T	F	T	F	T
T	T	F	F	F	T	F
T	F	T	T	F	T	F
T	F	F	T	T	F	T
F	T	T	T	F	F	F
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

9. Prove or disprove: $\forall x \in \mathbb{Q}$, if $x \notin \mathbb{R}$ then $\sqrt{x} \in \mathbb{R}$.

The statement is true. Let $x \in \mathbb{Q}$ be arbitrary. Since every rational number is real, we must have $x \in \mathbb{R}$ as well. Hence $x \notin \mathbb{R}$ is FALSE, so the implication $(x \notin \mathbb{R}) \to (\sqrt{x} \in \mathbb{R})$ is true vacuously.

10. Prove the proposition: $\forall x \in \mathbb{N},\ \forall y \in \mathbb{N},\ (x < y) \to (\exists z \in \mathbb{N},\ x^2 < z < y^2)$.

Let $x, y \in \mathbb{N}$ be arbitrary. We will proceed by direct proof. Assume that $x < y$. Hence $y - x \in \mathbb{N}_0$ and $y \neq x$. Hence, in particular $y - x \geq 1$. Also, since $x, y \in \mathbb{N}$, we must have $x \geq 1$ and $y \geq 1$. Adding, we get $y + x \geq 2$. We combine these two facts (perhaps using problem 4 of this exam), getting $(y - x)(y + x) \geq 2$. Hence $y^2 - x^2 \geq 2$.

We now pick $z = x^2 + 1$. We have $z - x^2 = (x^2 + 1) - x^2 = 1 \in \mathbb{N}_0$, so $x^2 \leq z$; also $x^2 \neq z$, so $x^2 < z$. We also have $y^2 - z = y^2 - (x^2 + 1) = y^2 - x^2 - 1$. Since $y^2 - x^2 \geq 2$, we must have $y^2 - x^2 - 1 \geq 2 - 1 = 1$. So again we have $y^2 - z \in \mathbb{N}_0$ and also $y^2 \neq z$, so $z < y^2$. Combining, we get $x^2 < z < y^2$.